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AN AVERAGING THEOREM FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH SMALL NONLINEARITIES

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Abstract. Consider nonlinear Schrödinger equations with small nonlinearities

$$\frac{d}{dt}u + i(-\triangle u + V(x)u) = \epsilon\mathcal{P}(\triangle u, \nabla u, u, x), \quad x \in \mathbb{T}^d. \tag{$*$}$$

Let $\{\zeta_1(x), \zeta_2(x), \dots\}$ be the L_2-basis formed by eigenfunctions of the operator $-\triangle + V(x)$. For any complex function $u(x)$, write it as $u(x) = \sum_{k\geqslant 1} v_k\zeta_k(x)$ and set $I_k(u) = \frac{1}{2}|v_k|^2$. Then for any solution $u(t,x)$ of the linear equation $(*)_{\epsilon=0}$ we have $I(u(t,\cdot)) = const$. In this work it is proved that if $(*)$ is well posed on time-intervals $t \lesssim \epsilon^{-1}$ and satisfies there some mild a-priori assumptions, then for any its solution $u^\epsilon(t,x)$, the limiting behavior of the curve $I(u^\epsilon(t,\cdot))$ on time intervals of order ϵ^{-1}, as $\epsilon \to 0$, can be uniquely characterized by solutions of a certain well-posed effective equation.

1. Introduction.

We consider the Schrödinger equation

$$\frac{d}{dt}u + i(-\triangle u + V(x)u) = 0, \quad x \in \mathbb{T}^d, \tag{1.1}$$

and its nonlinear perturbation:

$$\frac{d}{dt}u + i(-\triangle u + V(x)u) = \epsilon\mathcal{P}(\triangle u, \nabla u, u, x), \quad x \in \mathbb{T}^d, \tag{1.2}$$

where $\mathcal{P} : \mathbb{C}^{d+2}\times\mathbb{T}^d \to \mathbb{C}$ is a smooth function, $1 \leqslant V(x) \in C^n(\mathbb{T}^d)$ is a potential (we will assume that n is sufficiently large) and $\epsilon \in (0,1]$ is the perturbation parameter. For any $p \in \mathbb{R}$ denote by H^p the Sobolev space of complex-valued periodic functions, provided with the norm $||\cdot||_p$,

$$||u||_p^2 = \Big\langle (-\triangle)^p u,\ u\Big\rangle + \langle u,\ u\rangle, \quad \text{if} \quad p \in \mathbb{N},$$

where $\langle\cdot,\cdot\rangle$ is the real scalar product in $L^2(\mathbb{T}^d)$,

$$\langle u,\ v\rangle = Re\int_{\mathbb{T}^d} u\bar{v}dx, \quad u,\ v \in L^2(\mathbb{T}^d).$$

If $p > \frac{d}{2} + 2 = p_d$, then the mapping $H^p \to H^{p-2}$, $u(x) \mapsto \mathcal{P}(\triangle u, \nabla u, u, x)$ is smooth (see below Lemma 3.1). For any $T > 0$, a curve $u \in C([0,T], H^p)$, $p > p_d$, is called a solution of (1.2) in H^p if it is a mild solution of this equation. That is, if

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the relation obtained by integrating (1.2) in t from 0 to s holds for any $0 \leqslant s \leqslant T$. We wish to study long-time behaviours of solutions for (1.2) and assume:

Assumption A *(a-priori estimate). Fix some $T > 0$. For any $p > p_d + 2$, there exists $n_1(p) > 0$ such that if $n \geqslant n_1(p)$, then for any $0 < \epsilon \leqslant 1$, the perturbed equation (1.2), provided with initial data*

$$u(0) = u_0 \in H^p, \tag{1.3}$$

has a unique solution $u(t,x) \in H^p$ such that

$$||u||_p \leqslant C(T, p, ||u_0||_p), \quad \text{for} \quad t \in [0, T\epsilon^{-1}].$$

Here and below the constant C also depends on the potential $V(x)$.

Denote the operator

$$A_V u := -\triangle u + V(x)u.$$

Let $\{\zeta_k\}_{k\geqslant 1}$ and $\{\lambda_k\}_{k\geqslant 1}$ be its real eigenfunctions and eigenvalues, ordered in such a way that

$$1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots.$$

We say that a potential $V(x)$ is *non-resonant* if

$$\sum_{k=1}^{\infty} \lambda_k s_k \neq 0, \tag{1.4}$$

for every finite non-zero integer vector $(s_1, s_2, \cdots)$. For any complex-valued function $u(x) \in H^p$, we denote by

$$\Psi(u) := v = (v_1, v_2, \cdots), \quad v_j \in \mathbb{C}, \tag{1.5}$$

the vector of its Fourier coefficients with respect to the basis $\{\zeta_k\}$, i.e. $u(x) = \sum_{k=1}^{\infty} v_k \zeta_k$. In the space of complex sequences v, we introduce the norms

$$|v|_p^2 = \sum_{k\geqslant 1} |v_k|^2 \lambda_k^p, \quad p \in \mathbb{R},$$

and define $h^p := \{v : |v|_p < +\infty\}$. Denote

$$I_k = \frac{1}{2}|v_k|^2, \quad \varphi_k = \text{Arg } v_k, \quad k \geqslant 1. \tag{1.6}$$

Then $(I, \varphi) \in \mathbb{R}^\infty \times \mathbb{T}^\infty$ are the action-angles for the linear equation (1.1). That is, in these variables equation (1.1) takes the integrable form

$$\frac{d}{dt} I_k = 0, \quad \frac{d}{dt}\varphi_k = \lambda_k, \quad k \geqslant 1. \tag{1.7}$$

Abusing notation we will write $v = (I, \varphi)$. Define h_I^p to be the weighted l^1-space

$$h_I^p := \Big\{ I = (I_1, \dots) \in \mathbb{R}^\infty : |I|_p^\sim < +\infty \Big\}, \quad |I|_p^\sim = 2\sum_{i=1}^{\infty} \lambda_i^p |I_i|,$$

and consider the mapping

$$\pi_I : h^p \to h_I^p,\ v \mapsto I, \quad I_j(v) = \frac{1}{2}|v_j|^2, \quad j \geqslant 1.$$

It is continuous and its image is the positive octant $h_{I+}^p = \{I \in h_I^p : I_j \geqslant 0, \forall j\}$.

We mainly concern with the long time behavior of the actions $I(u(t)) \in \mathbb{R}_+^\infty$ of solutions for the perturbed equation (1.2) for $t \lesssim \epsilon^{-1}$. For this purpose, it is convenient to pass to the slow time $\tau = \epsilon t$ and write equation (1.2) in the action-angle coordinates (I, φ):

$$\dot{I}_k = F_k(I, \varphi), \quad \dot{\varphi}_k = \epsilon^{-1}\lambda_k + G_k(I, \varphi), \quad k \geqslant 1, \tag{1.8}$$

where $I \in \mathbb{R}^\infty$, $\varphi \in \mathbb{T}^\infty$ and $\mathbb{T}^\infty := \{(\theta_i)_{i\in\mathbb{N}} : \theta_i \in \mathbb{T}\}$ is the infinite-dimensional torus endowed with the Tikhonov toppology. The functions F_k and G_k, $k \geqslant 1$ represent the perturbation term $\mathcal{P}$, written in the action-angle coordinates. In the finite dimensional situation, the *averaging principle* is well established for perturbed integrable systems. The principle states that for equations

$$\frac{d}{dt}I = \epsilon f(I, \varphi), \quad \frac{d}{dt}\varphi = W(I) + \epsilon g(I, \varphi),$$

where $I \in \mathbb{R}^M$ and $\varphi \in \mathbb{T}^m$, on time intervals of order ϵ^{-1} the action components $I(t)$ can be well approximated by solutions of the following averaged equation:

$$\frac{d}{dt}J = \epsilon\langle f\rangle(J), \quad \langle f\rangle(J) = \int_{\mathbb{T}^m} f(J, \varphi)d\varphi. \tag{1.9}$$

This assertion has been justified under various non-degeneracy assumptions on the frequency vector W and the initial data $(I(0), \varphi(0))$ (see [12]). In this paper we want to prove a version of the averaging principle for the perturbed Schrödinger equation (1.2). We define a corresponding averaged equation for (1.8) as in (1.9):

$$\dot{J}_k = \langle F_k\rangle(J), \quad \langle F_k\rangle(J) = \int_{\mathbb{T}^\infty} F_k(J, \varphi)d\varphi, \quad k \geqslant 1, \tag{1.10}$$

where $d\varphi$ is the Haar measure on $\mathbb{T}^\infty$. But now, in difference with the finite-dimensional case, the well-posedness of equation (1.10) is not obvious, since the map $\langle F\rangle(I) = (\langle F_1\rangle(I), \dots)$ is unbounded and the functions $\langle F_k\rangle(I)$, $k \geqslant 1$, may be not Lipschitz with respect to I in h^p_{I+}. In [9], S. Kuksin observed that the averaged equation (1.10) may be lifted to a regular 'effective equation' on the variable $v \in h^p$, which transforms to (1.10) under the projection π_I. To derive an effective equation, corresponding to our problem, we first use mapping Ψ to write (1.2) as a system of equation on the vector $v(\tau)$:

$$\dot{v} = \epsilon^{-1}d\Psi(u)(-iA_V(u)) + P(v). \tag{1.11}$$

Here $P(v)$ is the perturbation term $\mathcal{P}$, written in v-variables. This equation is singular when $\epsilon \to 0$. The effective equation for (1.11) is a certain regular equation

$$\dot{v} = R(v). \tag{1.12}$$

To define the effective vector filed $R(v)$, for any $\theta = (\theta_1, \theta_2, \cdots) \in \mathbb{T}^\infty$ let us denote by Φ_θ the linear operator in the space of complex sequences $(v_1, v_2, \cdots) \in h^p$ which multiplies each component v_j with $e^{i\theta_j}$. Rotation Φ_θ acts on vector fields on the v-space, and $R(v)$ is the result of action of Φ_θ on $P(v)$, averaged in θ:

$$R(v) = \int_{\mathbb{T}^\infty} \Phi_{-\theta}P(\Phi_\theta v)d\theta.$$

The map $R(v)$ is smooth with respect to v in h^p. Again, we understand solutions for equation (1.12) in the mild sense.

We now make the second assumption:

Assumption B *(local well-posedness of the effective equation). For any $p > p_d + 2$, there exists $n_2(p) > 0$ such that if $n \geqslant n_2(p)$, then for any initial data $v_0 \in h^p$, there exists $T(|v_0|_p) > 0$ such that the effective equations (1.12) has a unique solution*

$v \in C([0, T(|v_0|_p)], h^p)$. Here $T : \mathbb{R}_+ \to \mathbb{R}_{>0}$ is an upper semi-continuous function.

The main result of this paper is the following statement, where $v^\epsilon(\tau)$ is the Fourier transform of a solution $u^\epsilon(t, x)$ for the problem (1.2), (1.3) (existing by Assumption A), written in the slow time $\tau = \epsilon t$:

$$v^\epsilon(\tau) = \Psi\big(u^\epsilon(\epsilon^{-1}\tau)\big), \quad \tau \in [0, T].$$

We also assume Assumption B.

Theorem 1.1. *For any $p > p_d + 2$, if $n \geqslant \max\{p, n_1(p), n_2(p)\}$, then there exists $I^0(\cdot) \in C([0, T], h_I^p)$ such that for every $q < p$,*

$$I(v^\epsilon(\cdot)) \underset{\epsilon \to 0}{\longrightarrow} I^0(\cdot) \quad in \quad C([0, T], h_I^q).$$

Moreover $I^0(\tau)$, $\tau \in [0, T]$, solves the averaged equation (1.10) with initial data $I^0(0) = I(\Psi(u_0))$, and it may be written as $I^0(\tau) = I(v(\tau))$, where $v(\cdot)$ is the unique solution of the effective equation (1.12), equal to $\Psi(u_0)$ at $\tau = 0$.

Proposition 1.2. *The assumptions A and B hold if (1.2) is a complex Ginzburg-Landau equation*

$$\dot{u} + \epsilon^{-1} i(\triangle u + V(x)u) = \triangle u - \gamma_R f_p(|u|^2)u - i\gamma_I f_q(|u|^2)u, \quad x \in \mathbb{T}^d, \tag{1.13}$$

where the constants γ_R, γ_I satisfy

$$\gamma_R,\ \gamma_I > 0, \tag{1.14}$$

the functions $f_p(r)$ and $f_q(r)$ are the monomials $|r|^p$ and $|r|^q$, smoothed out near zero, and

$$0 \leqslant p, q < \infty \quad if \quad d = 1, 2 \quad and \quad 0 \leqslant p, q < \min\{\frac{d}{2}, \frac{2}{d-2}\} \quad if \quad d \geqslant 3. \tag{1.15}$$

This work is a continuation of the research started in [7], where the author proved a similar averaging principle (not for all but for typical initial data) for a perturbed KdV equation:

$$u_t + u_{xxx} - 6uu_x = \epsilon f(u)(x),\ x \in \mathbb{T},\ \int_{\mathbb{T}} u(t, x)dx = 0, \tag{1.16}$$

assuming the perturbation $\epsilon f(u)(\cdot)$ defines a smoothing mapping $u(\cdot) \mapsto f(u)(\cdot)$. This additional assumption is necessary to guarantee the existence of an quasi-invariant measure for the perturbed equation (1.16), which plays an essential role in the proof due to the non-linear nature of the unperturbed equation. Since in the present paper we deal with perturbations of a linear equation, this restriction is not needed.

In [10], a result similar to Theorem 0.1 was proved for weakly nonlinear stochastic CGL equation (1.13). There are many works on long-time behaviors of solutions for nonlinear Schrödinger equations. E.g. the averaging principle was justified in [8] for solutions of Hamiltonian perturbations of (1.1), provided that the potential $V(x)$ is non-degenerated and that the initial data $u_0(x)$ is a sum of finitely many Fourier modes. Several long-time stability theorems which are applicable to small amplitude solutions of nonlinear Schrödinger equations were presented in [1, 13, 6, 3]. The results in these works describe the dynamics over a time scale much longer than the $\mathcal{O}(\epsilon^{-1})$ that we consider, precisely, over a time interval of order ϵ^{-m}, with arbitrary m (even of order $\exp \epsilon^{-\delta}$ with $\delta > 0$ in [1, 13, 6]). These results are obtained under the assumption that the frequencies are completely resonant or highly non-resonant (Diophantine-type), by using the normal form techniques near an equilibrium (this is the reason for which they only apply to small amplitude solutions). See [2] and references therein for general theory of normal form for PDEs. In difference with the mentioned works, the research in this paper is based on the classical averaging method for finite dimensional systems, characterizing by the existence of slow-fast variables. It deals with arbitrary solution of equation (1.2) with sufficiently smooth initial data. Also note that the non-resonance assumption (1.4) is significantly weaker than those in the mentioned works.

Plan of the paper. In Section 2 we recall some spectral properties of the operator A_V. Section 3 is about the action-angle form of the perturbed linear Schrödinger equation (1.2). In Section 4 we introduce the averaged equation and the corresponding effective equation. Theorem 1.1 and Proposition 1.2 are proved in Section 5 and Section 6.

2. Spectral properties of A_V.

As in the introduction, $A_V = -\triangle + V(x)$, $x \in \mathbb{T}^d$, where $1 \leqslant V(x) \in C^n(\mathbb{T}^d)$ and $\{\lambda_k\}_{k\geqslant 1}$ are the eigenvalues of A_V. According to Weyl's law, the λ_k, $k \geqslant 1$, satisfy the following asympototics

$$\lambda_k = C_d k^{2/d} + o(k^{2/d}), \quad k \geqslant 1,$$

Fix an L^2-orthogonal basis of eigenfunctions $\{\zeta_k\}_{k\geqslant 1}$ corresponding to the eigenvalues $\{\lambda_k\}_{k\geqslant 1}$, and define the linear mapping Ψ as (1.5). For any $m \in \mathbb{N}$, we have $\langle A_V^m u, u\rangle = |v|_m^2$, where $v = \Psi u$. Noting that $\langle A_V^m u, u\rangle$ is equivalent to $||u||_m^2$ for $m = 1, \ldots, n$, since $V(x)$ is C^n-smooth, we have the following:

Lemma 2.1. *For every integer $p \in [0, n]$ the linear mapping $\Psi : H^p \to h^p$ is an isomorphism.*

We denote

$$C^n_{+1}(\mathbb{T}^d) := \{V(x) \geqslant 1 : V(x) \in C^n(\mathbb{T}^d)\}.$$

For any finite $M \in \mathbb{N}$ consider the mapping

$$\Lambda^M : C^n_{+1}(\mathbb{T}^d) \to \mathbb{R}^M, \quad V(x) \to (\lambda_1, \cdots, \lambda_M),$$

and define the open domain $E_M \subset C^n_{+1}(\mathbb{T}^d)$,

$$E_M := \{V | \lambda_1 < \lambda_2 < \cdots < \lambda_M\}.$$

The complement of E_M is a real analytic variety in $C^n(\mathbb{T}^d)$ of codimension at least 2, so E_M is connected. The mapping Λ^M is analytic in E_M (see [8]).

Let μ be a Gaussian measure with a non-degenerate correlation operator, supported by the space $C^n(\mathbb{T}^d)$ (see [4]). Then $\mu(C^n_{+1}(\mathbb{T}^d)) > 0$. Fix $s \in \mathbb{Z}^M \setminus \{0\}$. The set

$$Q_s := \{V \in E_M | \Lambda^M(V) \cdot s = 0\},$$

is closed in E_M. Since $\Lambda^M(V) \cdot s \not\equiv 0$ on E_M (e.g. see [8]), then $\mu(Q_s) = 0$ (see chapter 9 in [4] and the note [5]). Since this is true for any M and s as above, then we have:

Proposition 2.2. *The non-resonant potentials form a subset of $C^n_{+1}(\mathbb{T}^d)$ of full μ-measure.*

3. **Equation (1.2) in action-angle variables.** For $k = 1, 2, \dots,$ we denote:

$$\Psi_k : H^p \to \mathbb{C}, \quad \Psi_k(u) = v_k,$$

(see (1.5)). Let $u(t)$ be a solution of equation (1.2). Passing to slow time $\tau = \epsilon t$, we get for $v_k = \Psi_k(u(\tau))$ equations

$$\dot{v}_k + i\epsilon^{-1}\lambda_k v_k = \Psi_k(\mathcal{P}(\triangle u, \nabla u, u, x)), \quad k \geqslant 1. \tag{3.1}$$

Since $I_k(v) = \frac{1}{2}|\Psi_k|^2$ is an integral of motion for the Schrödinger equation (1.1), we have

$$\dot{I}_k = (\Psi_k(\mathcal{P}(\triangle u, \nabla u, u, x)), v_k) := F_k(v), \quad k \geqslant 1 \tag{3.2}$$

(Here and below $(\cdot,\cdot)$ indicates the real scalar product in $\mathbb{C}$, i.e. $(u, v) = Re\, u\bar{v}$.)

Denote $\varphi_k = \text{Arg } v_k$, if $v_k \neq 0$, and $\varphi_k = 0$, if $v_k = 0$, $k \geqslant 1$. Using equation (3.1), we get

$$\dot{\varphi}_k = \epsilon^{-1}\lambda_k + |v_k|^{-2}(\Psi_k(\mathcal{P}(\triangle u, \nabla u, u, x)), iv_k), \quad \text{if} \quad v_k \neq 0, \quad k \geqslant 1 \tag{3.3}$$

Denoting for brevity, the vector field in equation (3.3) by $\epsilon^{-1}\lambda_k + G_k(v)$, we rewrite the equation for the pair $(I_k, \varphi_k)(k \geqslant 1)$ as

$$\dot{I}_k = F_k(v) = F_k(I, \varphi), \quad \dot{\varphi}_k = \epsilon^{-1}\lambda_k + G_k(v). \tag{3.4}$$

(Note that the second equation has a singularity when $I_k = 0$.) We denote

$$F(I, \varphi) = (F_1(I, \varphi), F_2(I, \varphi), \cdots).$$

The following result is well known, see e.g. Section 5.5.3 in [14].

Lemma 3.1. *If $f(x) : \mathbb{C}^m \to \mathbb{C}^N$ is C^∞, then the mapping*

$$M_f : H^p(\mathbb{T}^d, \mathbb{C}^m) \to H^p(\mathbb{T}^d, \mathbb{C}^N), \quad u \mapsto f(u),$$

is C^∞-smooth for $p > d/2$. Moreover, it is bounded and Lipschitz, uniformly on bounded subsets of $H^p(\mathbb{T}^d, \mathbb{C}^m)$.

In the lemma below, P_k and P_k^j are some fixed continuous functions.

Lemma 3.2. *For any $j, k \in \mathbb{N}$, we have for any $p > p_d$*

(i)The function $F_k(v)$ is smooth in each space h^p.

(ii) For any $\delta > 0$, the function $G_k(v)\chi_{\{I_k \geqslant \delta\}}$ is bounded by $\delta^{-1/2}P_k(|v|_p)$.

(iii)For any $\delta > 0$, the function $\frac{\partial F_k}{\partial I_j}(I, \varphi)\chi_{\{I_j \geqslant \delta\}}$ is bounded by $\delta^{-1/2}P_k^j(|v|_p)$.

(iv) The function $\frac{\partial F_k}{\partial \varphi_j}(I, \varphi)$ is bounded by $P_k^j(|v|_p)$ and for any $m \in \mathbb{N}$ and any $(I_1, \cdots, I_m) \in \mathbb{R}_+^m$, the fucntion $F_k(I_1, \varphi_1, \cdots, I_m, \varphi_m, 0, \cdots)$ is smooth on $\mathbb{T}^m$.

Proof. Item (i) and (ii) follow directly from (3.2), (3.3), Lemmata 1.1 and 2.1. Item (iii) and (iv) follow directly from item (i) and the chain rule. □

Denote

$$\Pi_{I,\varphi} : h^p \to h_I^p \times \mathbb{T}^\infty, \quad \Pi_{I,\varphi}(v) = (I(v), \varphi(v)). \tag{3.5}$$

Definition 3.3. Let assumption A holds. Then for any $p \geqslant p_d + 2$ and $T > 0$, we call a curve $(I(\tau), \varphi(\tau))$, $\tau \in [0, T]$, a regular solution of equation (3.4), if there is a solution $u(t) \in H^p$ of equation (1.2) such that

$$\Pi_{I,\varphi}(\Psi(u(\epsilon^{-1}\tau))) = (I(\tau), \varphi(\tau)) \in h_I^p \times \mathbb{T}^\infty, \quad \tau \in [0, T].$$

Note that if $(I(\tau), \varphi(\tau))$ is a regular solution, then each $I_j(\tau)$ is a C^1-function, while $\varphi_j(\tau)$ may be discontinuous at points τ, where $I_j(\tau) = 0$.

For any $p \geqslant p_d + 2$, let $(I(\tau), \varphi(\tau))$ be a regular solution of (3.4) such that $|I(0)|_p \leqslant M_0$. Then by Assumption A, for any $\epsilon > 0$ and $T > 0$, we have

$$|I(\tau)|_p^{\sim} = \frac{1}{2}|v(p)|_p^2 \leqslant C(p, M_0, T), \quad \tau \in [0, T]. \tag{3.6}$$

4. **Averaged equation and effective equation.** For a function f on a Hilbert space H, we write $f \in Lip_{loc}(H)$ if

$$|f(u_1) - f(u_2)| \leqslant P(R)||u_1 - u_2||, \quad \text{if} \quad ||u_1||, ||u_2|| \leqslant R, \tag{4.1}$$

for a suitable continuous function P which depends on f. Clearly, the set of functions $Lip_{loc}(H)$ is an algebra. By Lemma 3.1,

$$F_k(v) \in Lip_{loc}(h^p), \quad k \in \mathbb{N}, \ p > p_d. \tag{4.2}$$

Let $f \in Lip_{loc}(h^p)$ and $v \in h^{p_1}$, where $p_1 > p$. Denoting by $\Pi^M, M \geqslant 1$ the projection

$$\Pi^M : h^0 \mapsto h^0, \quad (v_1, v_2, \cdots) \mapsto (v_1, \cdots, v_M, 0, \cdots),$$

we have

$$|v - \Pi^M v|_p \leqslant \lambda_M^{-(p_1-p)/2} |v|_{p_1}.$$

Accordingly,

$$|f(v) - f(\Pi^M v)| \leqslant P(|v|_p) \lambda_M^{-(p_1-p)/2} |v|_{p_1}. \tag{4.3}$$

We will denote $v^M = (v_1, \ldots, v_M)$ and identify v^M with $(v_1, \ldots, v_M, 0, \ldots)$ if needed. Similar notations will be used for vectors $\theta = (\theta_1, \theta_2, \ldots) \in \mathbb{T}^\infty$ and vectors $I = (I_1, \ldots) \in h_I^p$.

The torus $\mathbb{T}^M$ acts on the space $\Pi_M h^0$ by linear transformations Φ_{θ^M}, $\theta^M \in \mathbb{T}^M$, where $\Phi_{\theta^M} : (I^M, \varphi^M) \mapsto (I^M, \varphi^M + \theta^M)$. Similarly, the tous $\mathbb{T}^\infty$ acts on h^0 by linear transformations $\Phi_\theta : (I, \varphi) \mapsto (I, \varphi + \theta)$ with $\theta \in \mathbb{T}^\infty$.

For a function $f \in Lip_{loc}(h^p)$ and any positive integer N, we define the average of f in the first N angles as

$$\langle f \rangle_N(v) = \int_{\mathbb{T}^N} f\Big((\Phi_{\theta^N} \oplus \mathrm{id})(v)\Big) d\theta^N,$$

and define the averaging in all angles as

$$\langle f \rangle_\varphi(v) = \int_{\mathbb{T}^\infty} f(\Phi_\theta(v)) d\theta,$$

where $d\theta$ is the Haar measure on $\mathbb{T}^\infty$. We will denote $\langle \cdot \rangle_\varphi$ as $\langle \cdot \rangle$ when there is no confusion. The estimate (4.3) readily implies that

$$|\langle f \rangle_N(v) - \langle f \rangle(v)| \leqslant P(R) \lambda_N^{-(p_1-p)/2}, \quad \text{if} \quad |v|_{p_1} \leqslant R.$$

Let $v = (I, \varphi)$, then $\langle f \rangle_N$ is a function independent of $\varphi_1, \cdots, \varphi_N$, and $\langle f \rangle$ is independent of φ. Thus $\langle f \rangle$ can be written as $\langle f \rangle(I)$.

Lemma 4.1. *(See [11]). Let $f \in Lip_{loc}(h^p)$, then*

(i) Functions $\langle f \rangle_N(v)$ and $\langle f \rangle$ satisfy (4.1) with the same function P as f and take the same value at the origin.

(ii) They are smooth if f is. If f is C^∞-smooth, then for any M, $\langle f \rangle(I)$ is a smooth function of the first M components $I_1, \cdots, I_M$ of the vector I.

Proof. Item (i) and the first statement of item (ii) is obvious. Notice that $\langle f\rangle(v) = \langle f\rangle(\sqrt{I_1},\dots)$ is even on each variable $\sqrt{I_j}$, $j \geqslant 1$, i.e.

$$\langle f\rangle(\dots,-\sqrt{I_j},\dots) = \langle f\rangle(\dots,\sqrt{I_j},\dots), \quad j \geqslant 1.$$

Now the second statement of item (ii) follows from Whitney's theorem (see Lemma A in the Appendix). □

Denote $C^{0+1}(\mathbb{T}^n)$ the set of all Lipschitz functions on $\mathbb{T}^n$. The following result is a version of the classical Weyl theorem.

Lemma 4.2. *Let $f \in C^{0+1}(\mathbb{T}^n)$ for some $n \in \mathbb{N}$. For any non-resonant vector $\omega \in \mathbb{R}^n$ (see (1.4)) and any $\delta > 0$, there exists $T_0 > 0$ such that if $T \geqslant T_0$, $g \in C(\mathbb{T}^n)$ and $|g - f| \leqslant \delta/3$, then we have*

$$\Big|\frac{1}{T}\int_0^T g(x_0 + \omega t)dt - \langle g\rangle\Big| \leqslant \delta,$$

uniformly in $x_0 \in \mathbb{T}^n$.

Proof. It is well known that for any $\delta > 0$ and non-resonant vector $\omega \in \mathbb{R}^n$, there exists $T_0 > 0$ such that

$$\Big|\frac{1}{T}\int_0^T f(x_0 + \omega t)dt - \langle f\rangle\Big| \leqslant \delta/3, \quad \forall T \geqslant T_0,$$

(see e.g. Lemma 2.2 in [7]). Therefore if $T \geqslant T_0$, $g \in C(\mathbb{T}^n)$ and $|g - f| \leqslant \delta/3$, then

$$\begin{aligned}\Big|\frac{1}{T}\int_0^T g(x_0 + \omega t)dt - \langle g\rangle\Big| &\leqslant \Big|\frac{1}{T}\int_0^T f(x_0 + \omega t)dt - \langle f\rangle\Big| \\ &\quad + \frac{1}{T}\int_0^T |f(x_0 + \omega t) - g(x_0 + \omega t)|dt + |\langle f\rangle - \langle g\rangle| \leqslant \delta.\end{aligned}$$

This finishes the proof of the lemma. □

We denote $P_k(v) = \Psi_k(\mathcal{P}(\triangle u, \nabla u, u, x))|_{u=\Psi^{-1}v}$, then equations (3.4) becomes

$$\dot{I}_k = (v_k, P_k(v)), \quad \dot{\varphi}_k = \epsilon^{-1}\lambda_k + G_k(v), \quad k \geqslant 1. \tag{4.4}$$

The averaged equations have the form

$$\dot{J}_k = \langle (v_k, P_k)\rangle_\varphi(J), \quad k \geqslant 1, \tag{4.5}$$

i.e.

$$\langle (v_k, P_k)\rangle_\varphi = \int_{\mathbb{T}^\infty} (v_k e^{i\theta_k}, P_k(\Phi_\theta v))d\theta = (v_k, R_k(v)), \tag{4.6}$$

with

$$R_k(v) = \int_{\mathbb{T}} \Phi_{-\theta_k} P_k(\Phi_\theta)d\theta. \tag{4.7}$$

Similar to equation (1.2), for any $T > 0$, we call a curve $J \in C([0,T], h_I^p)$ a solution of equation (4.5) if for every $s \in [0,T]$ it satisfies the relation, obtained by integrating (4.5).

Consider the differential equations

$$\dot{v}_k = R_k(v), \quad k \geqslant 1. \tag{4.8}$$

Solutions of this system are defined similar to that of (1.2) and (4.5). Relation (4.6) implies:

Lemma 4.3. *If $v(\cdot)$ satisfies (4.8), then $I(v)$ satisfies (4.5).*

Following [9], we call equations (4.8) the *effective equation* for the perturbed equation (1.2).

Proposition 4.4. *The effective equation is invariant under the rotation Φ_θ. That is, if $v(\tau)$ is a solution of (4.8), then for each $\theta \in \mathbb{T}^\infty$, $\Phi_\theta v(\tau)$ also is a solution.*

Proof. Applying Φ_θ to (4.8) we get that

$$\frac{d}{d\tau}\Phi_\theta v = \Phi_\theta R(v).$$

Relation (4.7) implies that operations R and Φ_θ commute. Therefore

$$\frac{d}{d\tau}\Phi_\theta v = R(\Phi_\theta v).$$

The assertion follows. □

5. **Proof of the averaging theorem.** In this section we prove the Theorem 1.1 by studying the behavior of regular solutions of equation (3.4). We fix $p \geqslant p_d + 2$, assume $n \geqslant \max\{p, n_1(p), n_2(p)\}$ and consider $u_0 \in H^p$. So

$$\Pi_{I,\varphi}(\Psi(u_0)) = (I_0, \varphi_0) \in h^p_{I+} \times \mathbb{T}^\infty. \tag{5.1}$$

We denote

$$B_p(M) = \{I \in h^p_{I+} : |I|^\sim_p \leqslant M\}. \tag{5.2}$$

Without loss of generality, we assume $T = 1$. Fix any $M_0 > 0$. Let

$$(I_0, \varphi_0) \in B_p(M_0) \times \mathbb{T}^\infty := \Gamma_0,$$

and let $(I(\tau), \varphi(\tau))$ be a regular solution of system (3.4) with $(I(0), \varphi(0)) = (I_0, \varphi_0)$. Then by (3.6), there exists $M_1 \geqslant M_0$ such that

$$I(\tau) \in B_p(M_1), \quad \tau \in [0, 1]. \tag{5.3}$$

All constants below depend on M_1 (i.e. on M_0), and usually this dependence is not indicated. From the definition of the perturbation and Lemma 2.1 we know that

$$|\mathcal{F}(I, \varphi)|^\sim_{p-2} \leqslant C_{M_1}, \quad \forall (I, \varphi) \in B_p(M_1) \times \mathbb{T}^\infty. \tag{5.4}$$

Recall that we identify $I^m = (I_1, \dots, I_m)$ with $(I_1, \dots, I_m, 0, \dots)$, etc.

Fix any $n_0 \in \mathbb{N}$. By (4.2), for every $\rho > 0$, there is $m_0 \in \mathbb{N}$, depending only on n_0, M_1 and ρ, such that if $m \geqslant m_0$, then

$$|F_k(I, \varphi) - F_k(I^m, \varphi^m)| \leqslant \rho, \quad \forall (I, \varphi) \in B_p(M_1) \times \mathbb{T}^\infty, \tag{5.5}$$

where $k = 1, \cdots, n_0$.

From now on, we always assume that $(I, \varphi) \in B_p(M_1) \times \mathbb{T}^\infty$.

Since $V(x)$ is non-resonant, then by Lemma 2.2 and Lemma 3.2, for any $\rho > 0$, there exists $T_0 = T_0(\rho, n_0) > 0$, such that for all $\varphi \in \mathbb{T}^\infty$ and $T \geqslant T_0$,

$$\Big|\frac{1}{T}\int_0^T F_k(I^{m_0}, \varphi^{m_0} + \Lambda^{m_0} t)dt - \langle F_k\rangle(I^{m_0})\Big| < \rho, \tag{5.6}$$

where $k = 1, \dots, n_0$. Due to Lemma 3.2, we have

$$
\begin{aligned}
&|G_j(I,\varphi)| \leqslant \frac{C_0(j, M_1)}{\sqrt{I_j}}, \quad \text{if} \quad I_j \neq 0,\\
&|\frac{\partial F_k}{\partial I_j}(I,\varphi)| \leqslant \frac{C_0(k, j, M_1)}{\sqrt{I_j}}, \quad \text{if} \quad I_j \neq 0,\\
&|\frac{\partial F_k}{\partial \varphi_j}(I,\varphi)| \leqslant C_0(k, j, M_1).
\end{aligned} \tag{5.7}
$$

From Lemma 3.1, we know

$$
|\langle F_k\rangle(I^{m_0}) - \langle F_k\rangle(\bar{I}^{m_0})| \leqslant C_1(k, m_0, M_1)|I^{m_0} - \bar{I}^{m_0}|, \tag{5.8}
$$

and by (4.2),

$$
|F_k(I^{m_0}, \varphi^{m_0}) - F_k(\bar{I}^{m_0}, \bar{\varphi}^{m_0})| \leqslant C_2(k, m_0, M_1)|v^{m_0} - \bar{v}^{m_0}|, \tag{5.9}
$$

where $\Pi_{I,\varphi}(v^{m_0}) = (I^{m_0}, \varphi^{m_0})$ (see (3.5)) and $|\cdot|$ is the l^∞-norm. Denote

$$
C_{M_1}^{n_0,m_0} = m_0 \cdot \max\{C_0, C_1, C_2 : 1 \leqslant j \leqslant m_0, 1 \leqslant k \leqslant n_0\}.
$$

From now on we shall use the slow time $\tau = \epsilon t$.

Lemma 5.1. *For $k = 1, \dots, n_0$, the I_k-component of any regular solution of (3.4) with initial data in Γ_0 can be written as:*

$$
I_k(\tau) = I_k(0) + \int_0^\tau \langle F_k\rangle(I(s))ds + \Xi(\tau),
$$

where for any $\gamma \in (0,1)$ the function $|\Xi(\tau)|$ is bounded on $[0,1]$ by

$$
\begin{aligned}
|\Xi(\tau)| \leqslant C_{M_1}^{n_0,m_0}\Big[&\frac{T_0\epsilon}{2\gamma^{1/2}} + \frac{T_0 C_{M_1}\epsilon}{2\gamma^{1/2}} + T_0 C_{M_1}\epsilon\\
&+ 4(\gamma + T_0 C_{M_1}\epsilon)^{1/2}\Big](\epsilon T_0 + 1) + 3\rho + 3\epsilon C_{M_1} T_0 \quad \tau \in [0,1],
\end{aligned} \tag{5.10}
$$

where $\rho > 0$ is arbitary and $T_0 = T_0(\rho, n_0)$ is as (5.6).

Proof. Let us divide the time interval $[0,\tau]$, $\tau \leqslant 1$, into subinterval $[a_i, a_{i+1}]$, $0 \leqslant i \leqslant d_0$, such that

$$
a_0 = 0, a_{d_0} = \tau, \quad a_{d_0} - a_{d_0-1} \leqslant \epsilon T_0,
$$

and $a_{i+1} - a_i = \epsilon T_0$, for $0 \leqslant i \leqslant d_0 - 2$. Then $d_0 \leqslant (T_0\epsilon)^{-1} + 1$. For each interval $[a_i, a_{i+1}]$ we define a subset $\Omega(i) \subset \{1, 2, \cdots, m_0\}$ in the following way:

$$
l \in \Omega(i) \quad \Longleftrightarrow \quad \exists \tau \in [a_i, a_{i+1}], \quad I_l(\tau) < \gamma.
$$

Then if $l \in \Omega(i)$, by (5.4) we have

$$
|I_l(\tau)| < T_0 C_{M_1}\epsilon + \gamma, \quad \tau \in [a_i, a_{i+1}].
$$

For $I = (I_1, I_2, \cdots)$ and $\varphi = (\varphi_1, \varphi_2, \cdots)$ we set

$$
\kappa_i(I) = \hat{I}, \quad \kappa_i(\varphi) = \hat{\varphi},
$$

where the vectors $\hat{I}$ and $\hat{\varphi}$ are defined as follows:

$$
\text{If} \quad l \in \Omega(i), \quad \text{then} \quad \hat{I}_l = 0, \hat{\varphi}_l = 0, \quad \text{else} \quad \hat{I}_l = I_l,\ \hat{\varphi}_l = \varphi_l.
$$

We abbreviate $\kappa_i(I,\varphi) = (\kappa_i(I), \kappa_i(\varphi))$.

Below, $k = 1, \dots, n_0$.

Then on $[a_i, a_{i+1}]$, noting $|v^{m_0} - \kappa_i(v^{m_0}| = \sqrt{2}|I^{m_0} - \kappa_i(I^{m_0})|^{1/2}$, and using (5.9) we have

$$\begin{aligned}
&\int_{a_i}^{a_{i+1}} \Big| F_k\Big(I^{m_0}(s), \varphi^{m_0}(s)\Big) - F_k\Big(\kappa_i\big(I^{m_0}(s), \varphi^{m_0}(s)\big)\Big)\Big| ds \\
&\leqslant \int_{a_i}^{a_{i+1}} C_{M_1}^{n_0,m_0}\sqrt{2}\Big|I^{m_0}(s) - \kappa_i\Big(I^{m_0}(s)\Big)\Big|^{1/2} ds \\
&\leqslant \epsilon\sqrt{2}T_0 C_{M_1}^{n_0,m_0}(\gamma + T_0 C_{M_1}\epsilon)^{1/2}.
\end{aligned} \tag{5.11}$$

By (5.5), we have

$$\int_0^\tau F_k(I(s), \varphi(s))ds = \int_0^\tau F_k(I^{m_0}(s), \varphi^{m_0}(s))ds + \xi_1(\tau), \tag{5.12}$$

where $|\xi_1(\tau)| \leqslant \rho\tau$.

Proposition 1.

$$\int_0^\tau F_k\Big(I^{m_0}(s), \varphi^{m_0}(s)\Big)ds = \sum_{i=0}^{d_0} \int_{a_i}^{a_{i+1}} F_k\Big(I^{m_0}(a_i), \varphi^{m_0}(s)\Big)ds + \xi_2(\tau),$$

where

$$|\xi_2| \leqslant \frac{1}{2} C_{M_1}^{n_0,m_0}\Big[4\sqrt{2}(\gamma + T_0 C_{M_1}\epsilon)^{1/2} + \gamma^{-1/2}T_0 C_{M_1}\epsilon\Big](\epsilon T_0 + 1). \tag{5.13}$$

Proof. We may write $\xi_2(t)$ as

$$\xi_2(t) = \sum_{i=0}^{d_0-1} \int_{a_i}^{a_{i+1}} \Big[F_k\Big(I^{m_0}(s), \varphi^{m_0}(s)\Big) - F_k\Big(I^{m_0}(a_i), \varphi^{m_0}(s)\Big)\Big]ds := \sum_{i=0}^{d_0-1} \tilde{I}_i.$$

For each i, by (5.4) and (5.7) we have

$$\begin{aligned}
&\int_{a_i}^{a_{i+1}} |F_k(\kappa_i(I^{m_0}(s)), \varphi^{m_0}(s)) - F_k(\kappa_i(I^{m_0}(a_i), \varphi^{m_0}(s))|ds \\
&\leqslant \int_{a_i}^{a_{i+1}} \gamma^{-1/2} C_{M_1}^{n_0,m_0}|\kappa_i(I^{m_0}(s) - I^{m_0}(a_i))|ds \\
&\leqslant \frac{1}{2} C_{M_1}^{n_0,m_0} C_{M_1} T_0^2 \gamma^{-1/2}\epsilon^2.
\end{aligned} \tag{5.14}$$

Replacing the integrand $F_k(I^{m_0}, \varphi^{m_0})$ by $F_k(\kappa_i(I^{m_0}, \varphi^{m_0}))$, using (5.11) and (5.14), we have

$$\tilde{I}_i \leqslant \frac{1}{2} C_{M_1}^{n_0,m_0}[4\sqrt{2}\epsilon T_0(\gamma + T_0 C_{M_1}\epsilon)^{1/2} + \gamma^{-1/2}T_0^2 C_{M_1}\epsilon^2].$$

The inequality (5.13) follows. □

On each subsegment $[a_i, a_{i+1}]$, we now consider the unperturbed linear dynamics $\tilde{\varphi}_i(\tau)$ of the angles $\varphi^{m_0} \in \mathbb{T}^{m_0}$:

$$\tilde{\varphi}_i(\tau) = \varphi^{m_0}(a_i) + \epsilon^{-1}\Lambda^{m_0}(\tau - a_i) \in \mathbb{T}^{m_0}, \quad \tau \in [a_i, a_{i+1}].$$

Proposition 2.

$$\int_0^\tau F_k\Big(I^{m_0}(a_i), \varphi^{m_0}(s)\Big)ds = \sum_{i=0}^{d_0-1} \int_{a_i}^{a_{i+1}} F_k\Big(I^{m_0}(a_i), \tilde{\varphi}_i(s)\Big)ds + \xi_3(\tau),$$

where

$$|\xi_3(\tau)| \leqslant [2\sqrt{2}C_{M_1}^{n_0,m_0}(\gamma + T_0 C_{M_1}\epsilon)^{1/2} + \frac{T_0\epsilon}{2\gamma}(C_{M_1}^{n_0,m_0})^2](1+\epsilon T_0). \tag{5.15}$$

Proof. On each $[a_i, a_{i+1}]$, notice that

$$\int_{a_i}^{a_{i+1}} \Big|\kappa_i\Big(\varphi^{m_0}(s) - \tilde{\varphi}_i(s)\Big)\Big| ds \leqslant \int_{a_i}^{a_{i+1}} \int_{a_i}^{s} \Big|\kappa_i\Big(\epsilon G^{m_0}(I(s'), \varphi(s'))\Big)\Big| ds' ds$$
$$\leqslant \int_{a_i}^{a_{i+1}} \int_{a_i}^{s} C_{M_1}^{n_0,m_0}\epsilon\gamma^{-1/2} ds' ds \leqslant \frac{T_0^2\epsilon^2}{2\gamma^{1/2}} C_{M_1}^{n_0,m_0}.$$

Here the first inequality comes from equation (3.4), and using (5.7) we can get the second inequality. Therefore, using again (5.7), we have

$$\int_{a_i}^{a_{i+1}} \Big[F_k\Big(\kappa_i(I^{m_0}(a_i), \varphi^{m_0}(s))\Big) - F\Big(\kappa_i(I^{m_0}(a_i), \tilde{\varphi}_i(s))\Big)\Big] ds$$
$$\leqslant \int_{a_i}^{a_{i+1}} C_{M_1}^{n_0,m_0}\Big|\kappa_i\Big(\varphi^{m_0}(s) - \tilde{\varphi}_i(s)\Big)\Big| ds$$
$$\leqslant \frac{T_0^2\epsilon^2}{2\gamma^{1/2}}(C_{M_1}^{n_0,m_0})^2$$

Therefore (5.15) holds for the same reason as (5.13). □

We will now compare the integrals $\int_{a_i}^{a_{i+1}} F_k(I^{m_0}(a_i), \tilde{\varphi}_i(s))ds$ with the average values $\langle F_k(I^{m_0}(a_i))\rangle \epsilon T_0$.

Propositon 3.

$$\sum_{i=0}^{d_0-1} \int_{a_i}^{a_{i+1}} F_k\Big(I^{m_0}(a_i), \tilde{\varphi}_i(s)\Big) ds = \sum_{i=1}^{d_0-1} T_0 \langle F_k\rangle\Big(I^{m_0}(a_i)\Big) + \xi_4(\tau),$$

where

$$|\xi_4(\tau)| \leqslant \rho + 2C_{M_1}\epsilon T_0. \tag{5.16}$$

Proof. For $0 \leqslant i \leqslant d_0 - 2$, by (5.6)

$$\Big|\int_{a_i}^{a_{i+1}} \Big[F_k\Big(I^{m_0}(a_i), \tilde{\varphi}_i(s)\Big) - \langle F_k\rangle\Big(I^{m_0}(a_i)\Big)\Big] ds\Big| \leqslant \epsilon\rho T_0.$$

So

$$\sum_{i=0}^{d_0-2} \Big|\int_{a_i}^{a_{i+1}} F_k\Big(I^{m_0}(a_i), \tilde{\varphi}_i(s)\Big) ds - \langle F_k\rangle\Big(I^{m_0}(a_i)\Big) T_0\Big| \leqslant (d_0 - 1)\epsilon\rho T_0.$$

Moreover,

$$\Big|\int_{a_{d_0-1}}^{\tau} \Big[F_k\Big(I^{m_0}(a_i), \tilde{\varphi}_i(s)\Big) - \langle F_k\rangle\Big(I^{m_0}(a_i)\Big)\Big] ds\Big| \leqslant 2C_{M_1}\epsilon T_0.$$

This implies the inequality (5.16). □

Proposition 4.

$$\sum_{i=1}^{d_0-1} (a_{i+1} - a_i)\langle F_k\rangle\Big(I^{m_0}(a_i)\Big) = \int_0^{\tau} \langle F_k\rangle\Big(I^{m_0}(s)\Big) ds + \xi_5(\tau),$$

where

$$|\xi_5(\tau)| \leqslant \epsilon C_{M_1} C_{M_1}^{n_0,m_0} T_0(\epsilon T_0 + 1). \tag{5.17}$$

Proof. Indeed, as

$$|\xi_5(\tau)| = \Big| \int_0^\tau \Big[\langle F_k \rangle \Big(I^{m_0}(s) \Big) ds - \sum_{i=1}^{d_0-1} (a_{i+1} - a_i) \langle F_k \rangle \Big(I^{m_0}(a_i) \Big) \Big|,$$

then using (5.4) and (5.8) we get

$$\begin{aligned} |\xi_5(\tau)| &\leqslant \sum_{i=0}^{d_0-1} \int_{s(i,j)} C_{M_1}^{n_0,m_0} |I^{m_0}(s) - I^{m_0}(a_i)| ds \\ &\leqslant \epsilon^2 \sum_{i=0}^{d_0-1} C_{M_1} C_{M_1}^{n_0,m_0} (T_0)^2 \leqslant \epsilon C_{M_1} C_{M_1}^{n_0,m_0} T_0 (\epsilon T_0 + 1). \end{aligned}$$

□

Finally, we have obvious

Proposition 5.

$$\int_0^\tau \langle F_k \rangle \Big(I^{m_0}(s) \Big) ds = \int_0^\tau \langle F_k \rangle \Big(I(s) \Big) ds + \xi_6(\tau),$$

and $|\xi_6(\tau)|$ *is bounded by* $\rho\tau$.

Gathering the estimates in Propositions 1-5, we obtain

$$I_k(\tau) = I_k(0) + \int_0^\tau F_k \Big(I(s), \varphi(s) \Big) ds = I_k(0) + \int_0^\tau \langle F_k \rangle \Big(I(s) \Big) ds + \Xi(\tau),$$

where $\Xi(\tau)| \leqslant \sum_{i=1}^6 |\xi_i(\tau)|$ satisfies (5.10). Lemma 4.1 is proved. □

Corollary 1. *For any* $\bar\rho > 0$, *with a suitable choice of* ρ, γ *and* T_0, *the function* $|\Xi(\tau)|$ *in Lemma 4.1 can be made less than* $\bar\rho$, *if* ϵ *is small enough.*

Proof. We choose $\gamma = \epsilon^\alpha, T_0 = \epsilon^{-\sigma}, \rho = \frac{\bar\rho}{9}$ with

$$1 - \alpha/2 - \sigma > 0, \ 0 < \sigma < 1.$$

Then for ϵ small enough, we have $|\Xi(\tau)| < \bar\rho$. □

For any $(I_0, \varphi_0) \in \Gamma_0$, let the curve $(I^\epsilon(\tau), \varphi^\epsilon(\tau)) \in h_I^p \times \mathbb{T}^\infty$, $\tau \in [0,1]$, be a regular solution of the equation (4.4) such that $(I^\epsilon(0), \varphi^\epsilon(0)) = (I_0, \varphi_0)$.

Lemma 5.2. *The family of curves* $\{I^\epsilon(\tau), \tau \in [0,1]\}_{0<\epsilon<1}$ *is pre-compact in* $C([0,1], h_I^{p-2})$. *Moreover every limiting (as* $\epsilon \to 0$*) curve* $I^0(\tau)$, $\tau \in [0,1]$ *is a solution of the averaged equation (4.5), satisfying*

$$|I^0(\tau)|_p^\sim \leqslant M_1, \quad \tau \in [0,1].$$

Proof. Due to (3.6) and (5.4), we know that for any $\epsilon \in (0,1)$,

$$|I^\epsilon(\tau)|_p^\sim \leqslant M_1, \quad |\frac{d}{d\tau} I(\tau)|_{p-2}^\sim \leqslant C_{M_1}, \quad \tau \in [0,1].$$

Then by the Arzelà-Ascoli theorem, we have that the set $\mathcal{I} := \{I^\epsilon(\tau), \tau \in [0,1]\}_{0<\epsilon<1}$ is pre-compact in $C([0,1], h_I^{p-2})$. Let $\{\rho_m\}_{m\in\mathbb{N}}$ be a sequence such that $\rho_m \searrow 0$. From Lemma 4.1 and Corollary 4.2, there is $\epsilon_m > 0$ such that if $\epsilon \leqslant \epsilon_m$, then for $k = 1, \ldots, m$, we have

$$\begin{aligned} &I_k^\epsilon(\tau) = I_k^\epsilon(0) + \int_0^\tau \langle F_k \rangle (I^\epsilon(s)) ds + \Xi_k(\tau), \\ &|\Xi_k(\tau)| \leqslant \rho_m, \quad \tau \in [0,1]. \end{aligned} \tag{5.18}$$

Let $I^0 = I^0(\tau)$, $\tau \in [0,1]$ be a limiting curve of the set $\mathcal{I}$ as $\epsilon \to 0$. Then we have

$$I^0 \in C([0,1], h_I^{p-2}) \quad \text{and} \quad |I^0(\tau)|_p^\sim \leqslant M_1, \quad \tau \in [0,1].$$

By (5.18), the curve $I^0(\cdot)$ solves the averaged equation (4.5). □

For any $\theta \in \mathbb{T}^\infty$ and any vector $I \in h^p_{I+}$ we set

$$V_\theta(I) = (V_{\theta_1}(I_1), V_{\theta_2}(I_2), \dots),$$

where $\theta = (\theta_1, \theta_2, \dots)$ and $V_{\theta_j}(I_j) = \sqrt{2I_j}\cos(\theta_j) + i\sqrt{2I_j}\sin(\theta_j)$, for every $j \geqslant 1$. Then $\varphi_j(V_{\theta_j}) \equiv \theta_j$, and for each $\theta \in \mathbb{T}^\infty$ the map $I \to V_\theta(I)$ is a right inverse of the map $v \to I(v)$. For any vector I we denote

$$I^{>N} = (I_{N+1}, I_{N+2}, \dots), \quad V_\theta^{>N}(I) = (V_{\theta_{N+1}}(I_{N+1}), V_{\theta_{N+2}}(I_{N+2}), \dots).$$

Lemma 5.3. *(Lifting) Let $I^0(\tau) = (I_k^0(\tau), k \geqslant 1) \in h^p_{I+}$, $\tau \in [0,1]$, be a solution of the averaged equation (4.5), constructed in Lemma 4.3. Then, for any $\theta \in \mathbb{T}^\infty$, there is a solution $v(\cdot)$ of the effective equation (4.8) such that*

$$I(v(\tau)) = I^0(\tau), \quad \tau \in [0,1], \quad \textit{and} \quad v(0) = V_\theta(I^0(0)). \tag{5.19}$$

Proof. [1] For any $m \in \mathbb{N}$, consider the non-autonomous finite dimensional systems

$$\dot{I}_k = \langle F_k \rangle \Big(I_1, \cdots, I_m, \Big(I^0(\tau)\Big)^{>m} \Big), \quad k = 1, \cdots, m, \tag{5.20}$$

$$\dot{v}_k = R_k \Big(v_1, \dots, v_m, V_\theta^{>m}(I^0(\tau)) \Big), \quad k = 1, \dots, m. \tag{5.21}$$

Obviously, $(I_1^0(\tau), \dots, I_m^0(\tau))$, $\tau \in [0,1]$ solves system (5.20). It is its unique solution with initial data $(I_1^0(0), \dots, I_m^0(0))$, since by Lemma 3.1 $\langle F_k \rangle$ is smooth with respect to the variables $(I_1, \dots, I_m)$.

For $\bar{v}_0 = (V_{\theta_1}(I_1^0(0)), \dots, V_{\theta_m}(I_m^0(0)))$, system (5.21) has a unique solution $v^m(\tau)$, defined for $\tau \in [0, T')$, with $v^m(0) = \bar{v}_0$, where $T' \leqslant 1$ and $v^m(\tau) \xrightarrow{\tau \to T'} \infty$ if $T' < 1$. Due to equality (4.6), $I(v^m)(\tau)$ solves system (5.20) in time interval $[0, T')$. Since $I(v^m(0)) = (I_1^0(0), \cdots, I_m^0(0))$, therefore $T' = 1$ and

$$I(v^m(\tau)) \equiv (I_1^0(\tau), \dots, I_m^0(\tau)) \quad \text{for} \quad 0 \leqslant \tau \leqslant 1.$$

Now denote

$$V_m(\tau) = (v^m(\tau), V_\theta^{>m}(\tau)), \quad \tau \in [0,1].$$

For the same reason as in the proof of Lemma 4.3, the family $\{V_m(\tau), \tau \in [0,1]\}_{m \in \mathbb{N}}$ is pre-compact in $C([0,1], h^{p-2})$ and

$$V_m(0) = V_\theta(I^0(0)), \quad I(V_m(\tau)) = I^0(\tau), \quad \tau \in [0,1], \quad m \in \mathbb{N}.$$

So any limiting (as $m \to \infty$) curve $v(\cdot)$ of the family$\{V_m(\tau), \tau \in [0,1]\}_{m \in \mathbb{N}}$ is a solution of the effective equation (4.8), satisfying equalities (5.19). The lemma is proved. □

Lemma 5.4. *(uniqueness) Under the same assumptions of Lemma 4.3, we have $I^0(\cdot) \in C([0,1], h_I^p)$ and for every $q < p$,*

$$I^\epsilon(\cdot) \underset{\epsilon \to 0}{\longrightarrow} I^0(\cdot) \quad \textit{in} \quad C([0,1], h_I^q). \tag{5.22}$$

[1]This argument is a simplified version of the proof of Theorem 3.1 in [9]

Proof. Let $I^0(\cdot)$ and $J^0(\cdot)$ be two limiting curves of the family $\{I^\epsilon(\cdot)\}_{0<\epsilon<0}$, as $\epsilon \to 0$, in $C([0,1],h_I^{p-2})$. Then by Lemma 4.4, for any $\theta \in \mathbb{T}^\infty$, there are solutions $v_I(\cdot)$, $v_J(\cdot)$ of the effective equation (4.8) such that for $0 \leqslant \tau \leqslant 1$,

$$I(v_I(\tau)) = I^0(\tau), \quad I(v_J(\tau)) = J(\tau), \quad v_I(0) = v_J(0) = v_0 = V_\theta(I_0).$$

Due to assumption B, for initial data v_0 the effective equation (4.8) has a unique solution $v_E(\cdot) \in C\big(\big[0, T(|v_0|_p)\big), h^p\big)$. Therefore

$$v_I(\tau) = v_J(\tau) = v_E(\tau). \tag{5.23}$$

This relation holds for $\tau \leqslant 1$ if $T(|v_0|_p) > 1$ and for $\tau < T(|v_0|_p)$ if $T(|v_0|_p) < 1$. But if $T(|v_0|_p) < 1$, then $|v_E(\tau)|_p \to \infty$ as $\tau \to T(|v_0|_p)$. By the construction in Lemmata 4.3 and 4.4, we know $|v_I(\tau)|_p^2 \leqslant M_1$ for $\tau \in [0,1]$. Together with (5.23) we have that $T(|v_0|_p) > 1$. Hence $I^0 = J^0$, $I^0 \in C([0,1],h_I^p)$ and

$$I^\epsilon(\cdot) \underset{\epsilon\to 0}{\to} I^0(\cdot) \quad \text{in} \quad C([0,1],h_I^{p-2}). \tag{5.24}$$

For any $q < p$, assume that the convergence (5.22) do not holds. Then there exists $\delta > 0$ and sequences $\epsilon_n, \tau_n \in [0,1]$ such that

$$\epsilon_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad |I^{\epsilon_n}(\tau_n) - I^0(\tau_n)|_q^\sim \geqslant \delta. \tag{5.25}$$

Takes subsequence $\{n_k\}$ such that $\tau_{n_k} \to \tau_0$ as $n_k \to \infty$. Since the sequence $\{I^{\epsilon_{n_k}}(\tau_{n_k})\}$ is pre-compact in h_I^q, and by (5.24), its limiting point as $n_k \to \infty$ equals $I^0(\tau_0)$, so we have $I^{\epsilon_{n_k}}(\tau_{n_k})$ converges to $I^0(\tau_0)$ in h_I^q as n_k goes to ∞. This contradicts with (5.25). So we completes the proof of Lemma 4.5 and also the proof of Theorem 1.1. $\square$

6. **Application to complex Ginzburg-Landau equations.** In this section we prove that assumptions A and B hold for equation (1.13), satisfying (1.14) and (1.15).

6.1. **Verification of Assumption A.** In this subsection, we denote by $|\cdot|_s$ the L^s-norm. Let $u(\tau)$ be a solution of equation (1.13) such that $u(0,x) = u_0$. Then

$$\begin{aligned}\frac{d}{d\tau}||u(\tau)||_0^2 &= 2\langle u, \dot u\rangle = 2\langle u, -\epsilon^{-1} i A_V u + \triangle u - \gamma_R |u|^{2p}u - i\gamma_I |u|^{2q}u\rangle,\\ &= -2||u||_1^2 + 2||u||_0^2 - 2\gamma_R |u|_{2p+2}^{2p+2}.\end{aligned}$$

Since $||u||_0^2 \leqslant |u|_{2p+2}^2$, then relation $||u(\tau_1)||_0 > \gamma_R^{-1/2p} = B_2$ implies that

$$\frac{d}{d\tau}||u(\tau_1)||_0^2 < 0.$$

So for any $T > 0$ we have

$$||u(T)||_0 \leqslant \min\{B_2,\ e^T||u_0||_0\}. \tag{6.1}$$

Now we rewrite equation (1.13) as follows:

$$\dot u + \epsilon^{-1} i(\triangle u + V(x)u + \epsilon\gamma_I |u|^{2q}u) = \triangle u - \gamma_R |u|^{2p}u. \tag{6.2}$$

For any $k \in \mathbb{N}$, denote

$$||u||_k^{\sim 2} = \langle A_V^k u, u\rangle, \quad A_V = -\triangle + V(x).$$

The l.h.s is a hamiltonian system with the hamiltonian function $\epsilon^{-1}H(u)$,

$$H(u) = \frac{1}{2}\langle A_V u, u\rangle + \frac{\epsilon}{2q+2}|u|_{2q+2}^{2q+2}.$$

We have $dH(u)(v) = \langle A_V u, v\rangle + \epsilon\gamma_I \langle |u|^{2q}u, v\rangle$, and if v is the vector field in the l.h.s of (6.2), then $dH(u)(v) = 0$. So we have

$$\begin{aligned}\frac{d}{d\tau}H(u(\tau)) &= -\gamma_R\langle A_V u, |u|^{2p}u\rangle + \langle A_V u, \triangle u\rangle \\ &\quad - \epsilon\gamma_I\gamma_R|u|^{2p+2q+2}_{2p+2q+2} + \epsilon\gamma_I\langle |u|^{2q}u, \triangle u\rangle,\end{aligned}$$

Denoting $U_q(x) = \frac{1}{q+1}u^{q+1}$ and $U_p = \frac{1}{p+1}u^{p+1}$, we get

$$\langle |u|^{2q}u, \triangle u\rangle \leqslant -\int_{\mathbb{T}^n} |\nabla u|^2|u|^{2q}dx = -||\nabla U_q||_0^2,$$

and a similar relation holds for q replaced by p. Therefore

$$\begin{aligned}\frac{d}{d\tau}H(u(\tau)) \leqslant& -\frac{1}{2}||u||_2^2 - \gamma_R||\nabla U_p||_0^2 - \epsilon\gamma_I||\nabla U_q||_0^2 - \epsilon\gamma_I\gamma_R|u|^{2p+2q+2}_{2p+2q+2} \\ &- \int_{\mathbb{T}^d} V(x)|\nabla u|^2dx + C_1||u||_0^2,\end{aligned}$$

where C_1 depends only on $|V|_{C^2}$. By this relation and (6.1), we have

$$H(u(T)) \leqslant H(u(0)) + C_1TB_2^2, \quad \text{for any} \quad T > 0. \tag{6.3}$$

So

$$||u(T)||_1^{\sim 2} \leqslant 2H(u(0)) + 2C_1TB_2^2, \quad \text{for any} \quad T > 0. \tag{6.4}$$

Simple calculation shows that

$$A_V^2 u = (-\triangle)^2u - 2V\triangle u - \nabla V\cdot\nabla u + (V^2 - \triangle V)u.$$

We consider

$$\frac{d}{d\tau}\langle A_V^2 u, u\rangle = 2\langle A_V^2 u, \triangle u - \gamma_R|u|^{2p}u - i\gamma_I|u|^{2q}u.\rangle \tag{6.5}$$

By the interpolation and Young inequality, we have

$$\begin{aligned}\langle A_V^2 u, \triangle u\rangle &\leqslant -||u||_3^2 + C_1(|V|)||u||_2^2 + C_2(|V|_{C^1})||u||_1^2 + C_3(|V|_{C^2})||u||_0^2 \\ &\leqslant -||u||_3^2 + C_1||u||_3^{\frac{4}{3}}||u||_0^2 + C_2||u||_3^{\frac{2}{3}}||u||_0^2 + C_3||u||_0^2 \\ &\leqslant -\frac{3}{4}||u||_3^2 + C(|V|_{C^2}, ||u||_0).\end{aligned} \tag{6.6}$$

We deduce from integration by part and Hölder inequality that

$$-\langle(-\triangle)^2u, |u|^{2p}u\rangle \leqslant ||u||_3|(|u|^{2p}\nabla u)|_2 \leqslant ||u||_3|u|^{2q}_{2pq_1}|\nabla u|_{p_1}, \tag{6.7}$$

where $p_1, q_1 < \infty$ satisfy $1/p_1 + 1/q_1 = 1/2$. Let p_1 and q_1 have the form

$$p_1 = \frac{2d}{d-2s}, \quad q_1 = \frac{d}{s}.$$

We specify parameter s: For $d \geqslant 3$, choose $s = p(d-2) < \min\{d/2, 2\}$; for $d = 1, 2$, choose $s \in (0, \frac{1}{2})$. Due to condition (1.15), we have the Sobolev embeddings

$$H^s(\mathbb{T}^d) \to L^{p_1}(\mathbb{T}^d) \quad \text{and} \quad H^1(\mathbb{T}^d) \to L^{2pq_1}(\mathbb{T}^d),$$

implying that

$$|\nabla u|_{p_1} \leqslant ||u||_{1+s}, \quad |u|^{2p}_{2pq_1} \leqslant ||u||_1^{2p}.$$

Applying again the interpolation and Young inequality we find that for any $\delta > 0$,

$$\begin{aligned}-\langle \triangle^2 u, |u|^{2p}u\rangle &\leqslant ||u||_3||u||_{1+s}||u||_1^{2p}\\ &\leqslant C||u||_3^{1+\frac{1+s}{3}}||u||_0^{\frac{2-s}{3}}||u||_1^{2p} \\ &\leqslant \delta||u||_3^2 + C(\delta)(||u||_0^{\frac{2-s}{3}}||u||_1^{2p})^{\frac{2-s}{6}},\end{aligned}\tag{6.8}$$

We can deal with other terms in (6.5) and (6.7) similarly. With suitable choice of δ, from the inequality above together with (6.6), we can get that for any $T > 0$

$$||u(T)||_2^{\smile 2} + \int_0^T ||u||_3^2 d\tau \leqslant ||u(0)||_2^{\smile 2} + C(2, |V|_{C^4}, T, B_2),\tag{6.9}$$

By similar argument, for any $m \geqslant 3$ and $T > 0$ we can obtain

$$||u(T)||_m^{\smile 2} + \int_0^T ||u||_{m+1}^2 d\tau \leqslant ||u(0)||_m^{\smile 2} + C(m, |V|_{C^{4m}}, T, B_2),$$

Then

$$||u(T)||_m \leqslant C(||u(0)||_m, |V|_{C^{4m}}, m, T, B_2), \quad \text{for any } T > 0.$$

This finishes the verification of assumption A.

6.2. **Verification of Assumption B.** We follow [10]. In equation (1.13) with $u \in H^2$, we pass to the v-variable, $v = \Psi(u) \in h^2$:

$$\dot v_k + i\epsilon^{-1}\lambda_k = P_k(v), \quad k \geqslant 1.\tag{6.10}$$

Here

$$P_k = P_k^1 + P_k^2 + P_k^3,$$

where P^1, P^2 and P^3 are, correspondingly, the linear, nonlinear dissipative and nonlinear hamiltonian parts of the perturbation:

$$P^1(v) = \Psi(\triangle u), \quad P^2(v) = -\gamma_R\Psi(|u|^{2p}u), \quad P^3(v) = -i\gamma_I\Psi(|u|^{2q}u),$$

with $u = \Psi^{-1}(v)$. Following the procedure in Section 3, the effective equations for (1.13) has the form:

$$\dot v = \sum_{i=1}^3 R^i(v),\tag{6.11}$$

where

$$R^i(v) = \int_{\mathbb{T}^\infty} \Phi_{-\theta}P^i(\Phi_\theta)\Big)d\theta, \quad i = 1,2,3.$$

Consider the operator

$$\mathcal{L} := \Psi\circ(-\triangle)\circ\Psi^{-1} = \Psi\circ(A_V - V)\circ\Psi^{-1} := \hat A - \Psi\circ V\circ\Psi^{-1} := \hat A - \mathcal{L}^0.$$

Clearly, $\hat A$ is the diagonal operator $\hat A = \mathrm{diag}\{\lambda_j \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \; j \geqslant 1\}$. By Lemma 1.1, $\mathcal{L}^0 = \Psi\circ V\circ\Psi^{-1}$ defines bounded maps

$$\mathcal{L}^0 : \quad h^m \to h^m, \quad \forall m \leqslant n,$$

and in the space h^0 the operator $\mathcal{L}^0$ is self-adjoint. Since $\hat A$ commutes with the rotation Φ_θ, then

$$\begin{aligned}R^1 &= -\int_{\mathbb{T}^\infty}\Phi_{-\theta}\hat A\Phi_\theta v d\theta + \int_{\mathbb{T}^\infty}\Phi_{-\theta}\mathcal{L}^0(\Phi_\theta v)d\theta\\ &= -\hat A v + R^0(v), \quad R^0(v) = \int_{\mathbb{T}^\infty}\Phi_{-\theta}\mathcal{L}^0(\Phi_\theta v)d\theta.\end{aligned}\tag{6.12}$$

Since for $v = (v_1, v_2, \dots)$, we have

$$\mathcal{L}^0(v)_j = \sum_{i=0}^{+\infty} \langle V(x) v_i \varphi_i(x), \varphi_j(x)\rangle, \quad j \geqslant 1,$$

then,

$$R_k^0(v) = \sum_{j=1}^{+\infty} \int_{\mathbb{T}^\infty} \langle V(x) v_j e^{i\theta_j} \varphi_j(x), e^{i\theta_k} \varphi_k(x)\rangle d\theta = v_k \langle V\varphi_k, \varphi_k\rangle.$$

That is,

$$R^1 = \operatorname{diag}\{-\lambda_k + M_k, k \geqslant 1\}, \quad M_k = \langle V\varphi_k, \varphi_k\rangle. \tag{6.13}$$

The term $R^2(v)$ is defined as an integral with the integrand

$$\Phi_{-\theta} P^2 \Phi_\theta(v) = -\gamma_R \Phi_{-\theta} \Psi(f_p(|u|^2)u)|_{u=\Psi^{-1}\Phi_\theta v} := F_\theta(v).$$

Define $\mathcal{H}(u) = \int \mathcal{F}(|u|^2)dx$, where $\mathcal{F}' = \frac{1}{2} f_p$. Then $\nabla \mathcal{H}(u) = f_p(|u|^2)u$. Denoting $\Psi^{-1}\Phi_\theta = L_\theta$, we have

$$F_\theta(v) = -\gamma_R L_\theta^* \nabla \mathcal{H}(u)|_{u=L_\theta(v)} = -\gamma_R \nabla(\mathcal{H} \circ L_\theta(v)).$$

So

$$R^2(v) = -\gamma_R \nabla_v \Big(\int_{\mathbb{T}^\infty} (\mathcal{H} \circ \Psi^{-1})(\Psi_\theta v) d\theta \Big) = -\gamma_R \nabla_v \langle \mathcal{H} \circ \Psi^{-1}\rangle.$$

Similarly, we have $R^3 = -i\gamma_I \nabla_v \langle \mathcal{G} \circ \Psi^{-1}\rangle$ with $\nabla \mathcal{G}(u) = f_q(|u|^2)u$. Since $\langle \mathcal{G} \circ \Psi^{-1}\rangle$ is a function only of the action $(I_1, \dots)$, we have that $\nabla_{v_k} \langle \mathcal{G} \circ \Psi^{-1}\rangle$ is proportional to v_k. Then $v_k \cdot R_k^3(v) = 0$. That is, it contributes a zero term in the averaged equation. Hence we could set the effective equation to be

$$\dot{v} = R^1(v) + R^2(v).$$

It is a quasi-linear heat equation, written in Fourier coefficients, which is known to be locally well posed. This verifies assumption B.

Appendix. Consider the l_2-space of sequences $x = (x_1, x_2, \dots)$. The following lemma is a slight modification of the well known theorem of Whitney [15].

Lemma A. *For any $n \in \mathbb{N}$, let $f \in C^\infty(l_2)$ be even in n variables, i.e.*

$$f(x_1, \dots, x_i, \dots) = f(x_1, \dots, -x_i, \dots), \quad i = 1, 2, \dots, n.$$

Then there exists $g_n \in C^\infty(l_2)$ such that

$$g_n(x_1^2, \dots, x_n^2, x_{n+1}, \dots) = f(x_1, x_2, \dots).$$

Proof. For $n = 1$, we define $g_1(x_1, x_2, \dots) = f(x_1^{\frac{1}{2}}, x_2, \dots)$. Since f is even with respect to x_1, for any $s \in \mathbb{N}$, we have

$$f(x_1, x_2, \dots) = f(\hat{x}) + f_1(\hat{x})x_1^2 + \cdots + f_{s-1}(\hat{x})x_1^{2s-2} + \phi(x)x_1^{2s},$$

where $\hat{x} = (0, x_2, \dots)$, $f_i = [(2i)!]^{-1}\partial_{x_1}^{2i} f(\hat{x})$ and $\phi(x)$ is smooth when $x_1 \neq 0$, even with respect to x_1, and satisfies

$$\lim_{x_1 \to 0} x_1^k \partial_{x_1}^k \phi(x) = 0, \quad k = 1, \dots, 2s. \tag{A.1}$$

Set $\psi(x) = \phi(x_1^{\frac{1}{2}}, x_2, \dots)$, then

$$g_1(x) = f(\hat{x}) + f_1(\hat{x})x_1 + \cdots + f_{s-1}(\hat{x})x_1^{s-1} + \psi(x)x_1^s.$$

We wish to check that $g_1(x)$ is C^s-smooth with respect to x_1. It is sufficient to prove that the limits $\lim_{x_1 \to 0} x_1^k \partial_{x_1}^k \psi(x)$, $k = 1, \dots, s$, exist and are finite. Differentiating

$\psi(x_1^2, x_2, \dots) = \phi(x)$ with respect to x_1, we get that there are some constants a_{ki} such that

$$\partial_{x_1}^k \phi(x) = 2^k x_1^k \partial_{x_1}^k \psi(x_1^2, x_2, \dots) + \sum_{1 \leqslant i \leqslant k/2} a_{ki} x_1^{k-2i} \partial_{x_1}^{k-i} \psi(x_1^2, x_2, \dots), \quad k = 1, \dots, s.$$

Solving these equation successively for $x_1^{2k}\partial_{x_1}^k \psi$, $k = 1, \dots, s$, we obtain that there are some constant β_{ki} such that

$$x_1^{2k}\partial_{x_1}^k \psi(x_1^2, x_2, \dots) = \sum_{0 \leqslant i \leqslant k} \beta_{ki} x_1^{k-i} \partial_{x_1}^{k-i} \phi(x).$$

By $(A.1)$, we know the $\lim_{x_1 \to 0} x_1^k \partial_{x_1}^k \psi(x)$, $k = 1, \dots, s$, exist and are finite. So $g_1(x)$ is C^s -smooth. Since s is arbitrary and $g_1(x)$ defined in a unique way, we have $g_1 \in C^\infty(l^2)$ and $g_1(x_1^2, x_2, \dots) = f(x_1, x_2, \dots)$. This prove the statement of the lemma for $n = 1$.

For $n \geqslant 2$, the assertion of the lemma can be prove by induction. Assume we have proved the lemma for $m = n - 1$. Then there exists $g_{n-1} \in C^\infty(l_2)$ such that $g_{n-1}(x_1^2, \dots, x_{n-1}^2, x_n) = f(x_1, x_2, \dots)$ and g_{n-1} is even in variable x_n. Applying what we have proved for $m = 1$ to g_{n-1} with respect to x_n, we get the assertion for $m = n$. □

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